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A SHARP EXCEPTIONAL SET ESTIMATE FOR VISIBILITY

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Abstract. A Borel set $B \subset \mathbb{R}^n$ is *visible* from $x \in \mathbb{R}^n$, if the radial projection of B with base point x has positive $\mathcal{H}^{n-1}$ measure. I prove that if $\dim B > n-1$, then B is visible from every point $x \in \mathbb{R}^n \setminus E$, where E is an exceptional set with dimension $\dim E \leq 2(n-1) - \dim B$. This is the sharp bound for all $n \geq 2$.

Many parts of the proof were already contained in a recent previous paper by P. Mattila and the author, where a weaker bound for $\dim E$ was derived as a corollary from a certain slicing theorem. Here, no improvement to the slicing result is obtained; in brief, the main observation of the present paper is that the proof method gives the optimal result, when applied directly to the visibility problem.

1. Introduction

For $x \in \mathbb{R}^n$, let $\pi_x \colon \mathbb{R}^n \setminus \{x\} \to S^{n-1}$ be the radial projection

$$\pi_x(y) := \frac{y-x}{|y-x|}.$$

The following theorem is the main result of the paper:

Theorem 1.1. *Assume that $B \subset \mathbb{R}^n$ is a Borel set with $\dim B > n-1$. Then, there exists a set $E \subset \mathbb{R}^n$ with $\dim E \leq 2(n-1) - \dim B$ such that*

$$\mathcal{H}^{n-1}(\pi_x(B)) > 0, \qquad x \in \mathbb{R}^n \setminus E.$$

This is the sharp bound for every $n \geq 2$.

This settles a conjecture by P. Mattila and the author in [8], where it was proven that $\dim E \leq n-1$ as soon as $\dim B > n-1$. The same conjecture had earlier appeared in Mattila's survey paper [6], see (6.1) on p. 36.[1] Finally, the proof in the present paper also fills a small gap in the argument in [8], see the footnote on page 3.

Remark 1.2. The strict inequality $\dim B > n-1$ is necessary. In fact, if $B \subset \mathbb{R}^n$ is purely $(n-1)$-unrectifiable with $0 < \mathcal{H}^{n-1}(B) < \infty$, then it follows easily from the Besicovitch-Federer projection theorem that $\mathcal{H}^{n-1}(\pi_x(B)) = 0$ for almost every $x \in \mathbb{R}^n$. A more precise result is due to Marstrand [4], Theorem VI: if $B \subset \mathbb{R}^2$ is purely 1-unrectifiable with $0 < \mathcal{H}^1(B) < \infty$, then $\mathcal{H}^1(\pi_x(B)) = 0$ for all $x \in \mathbb{R}^2 \setminus E$, where $\dim E \leq 1$. It

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[1]In fact, (6.1) in [6] states the planar version of Theorem 1.1 not as a conjecture, but a *fact*, which should follow from Peres and Schlag's work [10]. However, Theorem 7.3 in [10] gives the bound $3 - \dim B$ instead of $2 - \dim B$ in the plane.

is entertaining to note that when $\dim B > 1$, the same is true with "$\mathcal{H}^1(\pi_x(B)) = 0$" replaced by "$\mathcal{H}^1(\pi_x(B)) > 0$".

Marstrand's result can be further improved for self-similar sets: Simon and Solomyak [11] have shown that if $B \subset \mathbb{R}^2$ is a purely 1-unrectifiable self-similar set in the plane with $0 < \mathcal{H}^1(B) < \infty$, and satisfying the open set condition, then $\mathcal{H}^1(\pi_x(B)) = 0$ for every base point $x \in \mathbb{R}^2$. There is also a recent, more quantitative, version of this result by Bond, Łaba and Zahl [3].

Notation 1.3. The Grassmannian manifold of all $(n-1)$-dimensional subspaces of $\mathbb{R}^n$ is denoted by $G(n, n-1)$, and the Haar probability measure on $G(n, n-1)$ is denoted by $\gamma_{n,n-1}$. Given a plane $V \in G(n, n-1)$, the mapping $\pi_V \colon \mathbb{R}^n \to V$ is the orthogonal projection onto V. If μ is a Radon measure on $\mathbb{R}^n$, its push-forward under π_V is denoted by $\pi_{V\sharp}\mu$, so that

$$\pi_{V\sharp}\mu(B) = \mu(\pi_V^{-1}(B)), \qquad B \subset V.$$

For $a, b > 0$, we write $a \lesssim b$, if there exists a constant $C \geq 1$ such that $a \leq Cb$; the constant C may, without special mention, depend on various "fixed" parameters in the proof, such as the dimension of the ambient space, or that of B in Theorem 1.1.

For $0 \leq s \leq n$, the s-dimensional Hausdorff measure is denoted by $\mathcal{H}^s$. The notation dim stands for Hausdorff dimension. Finally, if μ is a Radon measure on $\mathbb{R}^n$ and $0 \leq s \leq n$, the s-energy of μ is denoted by $I_s(\mu)$, so that by definition

$$I_s(\mu) = \iint \frac{d\mu x \, d\mu y}{|x-y|^s}.$$

It is well-known that, see Theorem 3.10 in [7], that

$$I_s(\mu) = c(n,s) \int |\hat{\mu}(\xi)|^2 |\xi|^{s-n} \, d\xi, \qquad 0 < s < n. \tag{1.1}$$

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3. Proof of the main theorem

The first part of this section contains the proof of the bound $\dim E \leq 2(n-1) - \dim B$. The second, far shorter, part discusses the question of sharpness.

3.1. **Proof of the upper bound for** $\dim E$**.** It suffices to prove the theorem for compact sets B, because if $\dim\{x : \mathcal{H}^{n-1}(\pi_x(B)) = 0\} > 2(n-1) - \dim B$ for some Borel set B, then also $\dim\{x : \mathcal{H}^{n-1}(\pi_x(K)) = 0\} > 2(n-1) - \dim K$ for some compact set $K \subset B$ with $n - 1 < \dim K \leq \dim B$. So assume that B is compact. Then, the set $E := \{x \in \mathbb{R}^n : \mathcal{H}^{n-1}(\pi_x(B)) = 0\}$ is Borel, and we make the counter assumption that

$$2(n-1) - s < \dim E < n - 1$$

for some $n - 1 < s < \dim B$ (such an s can be found if $\dim E > 2(n-1) - \dim B$). We may further assume that E and B are disjoint; if this is not true to begin with, choose two disjoint closed balls B_1 and B_2 such that $\dim[B \cap B_1] > s$, and $2(n-1) - \dim[B \cap B_1] < \dim[E \cap B_2]$. Finally, fix t strictly between $2(n-1) - s$ and $\dim E$, and find compactly supported Borel probability measures μ and ν inside B and E, respectively, such that $I_s(\mu) < \infty$, and $I_t(\nu) < \infty$. Then $\mathcal{H}^{n-1}(\pi_x(\operatorname{spt}\mu)) = 0$ for every $x \in \operatorname{spt}\nu$; to simplify notation, we assume that $B = \operatorname{spt}\mu$ and $E = \operatorname{spt}\nu$.

We briefly discuss the meaning of the assumption $\mathcal{H}^{n-1}(\pi_x(B)) = 0$ for $x \in E$. If $L_{V,x}$ is the line perpendicular to $V \in G(n, n-1)$ and passing through $x \in \mathbb{R}^n$, another way to write $\mathcal{H}^{n-1}(\pi_x(B)) = 0$, $x \in E$, is the following:

$$\gamma_{n,n-1}(\{V : L_{V,x} \cap B \neq \emptyset\}) = 0, \qquad x \in E. \tag{3.1}$$

This is where we needed to know that B and E are disjoint. Using Fubini's theorem, (3.1) implies that

$$\nu(\{x : L_{V,x} \cap B \neq \emptyset\}) = 0 \tag{3.2}$$

for $\gamma_{n,n-1}$ almost every $V \in G(n, n-1)$.

For $\delta \in (0,1)$, let $\psi_\delta \colon \mathbb{R}^n \to [0,\infty)$ be a radial compactly supported approximate identity (thus $\psi_\delta = \delta^{-n}\psi(x/\delta)$, where ψ is non-negative, radial, supported on $B(0,1)$ and has integral one). Let $\mu_\delta := \mu * \psi_\delta$, and consider the function

$$V \mapsto f_\delta(V) := \int_V \pi_{V\sharp}\mu_\delta \, d\pi_{V\sharp}\nu, \qquad V \in G(n, n-1).$$

We will need to know that

(i) $\|f_\delta\|_{L^1(G(n,n-1))} \geq c$ for some constant $c > 0$ independent of $\delta \in (0,1)$, and
(ii) there exists $p > 1$ (depending on n, s and t only) such that $\|f_\delta\|_{L^p(G(n,n-1))} \leq C$, where $C < \infty$ is independent of $\delta \in (0,1)$.[2]

In fact, (i) is precisely (3.4) in [8], so we skip the details: in brief, applying the Parseval formula and integrating in polar coordinates, one can show that $\|f_\delta\|_{L^1}$ equals a constant times $\iint |x-y|^{-(n-1)} \, d\mu_\delta x \, d\nu x$, which is uniformly bounded from below for $\delta \in (0,1)$.

We then prove (ii). Write $s' := 2(n-1) - t < s$, and let σ be a measure on $G(n, n-1)$ satisfying the growth condition $\sigma(B(x,r)) \lesssim r^h$ for some

$$\max\{t, n-1+s'-s\} < h < n-1.$$

Write $\mu_V^\delta := \pi_{V\sharp}\mu_\delta$, and $\nu_V := \pi_{V\sharp}\nu$. Under the previous restrictions, it is known (see discussion below) that

$$\int \int_V |x|^{t-(n-1)} |\widehat{\nu_V}(x)|^2 \, d\mathcal{H}^{n-1}(x) \, d\sigma V = c(n,t) \int I_t(\nu_V) \, d\sigma V \lesssim I_t(\nu) < \infty \tag{3.3}$$

and

$$\int \int_V |x|^{s'-(n-1)} |\widehat{\mu_V^\delta}(x)|^2 \, d\mathcal{H}^{n-1}(x) \, d\sigma V \lesssim 1 + I_s(\mu_\delta) \lesssim 1 + I_s(\mu) < \infty. \tag{3.4}$$

The bound (3.3) in the plane is due to Kaufman [2], and the higher dimensional analogue we need can be found in a paper by Mattila, see Lemma 5.1 in [5]. As such, the bound (3.4) is most likely due to Peres and Schlag [10], but it is certainly inspired by earlier work of Falconer [1]; a proof can also be found on p. 81 in the book [7].

[2]This L^p-estimate was missing from the paper [8].

Armed with Parseval, (3.3), (3.4) and Cauchy-Schwarz, we make the following estimate:

$$
\begin{aligned}
\int \int_V \mu_V^\delta \, d\nu_V \, d\sigma V &\leq \int \int_V |\widehat{\mu_V^\delta}(x)||\widehat{\nu_V}(x)| \, d\mathcal{H}^{n-1}(x) \, d\sigma V \\
&= \int \int_V |x|^{(s'-(n-1))/2}|x|^{(t-(n-1))/2}|\widehat{\mu_V^\delta}(x)||\widehat{\nu_V}(x)| \, d\mathcal{H}^{n-1}(x) \, d\sigma V \\
&\leq \int \left(\int_V |x|^{t-(n-1)}|\widehat{\nu_V}(x)|^2 \, d\mathcal{H}^{n-1}(x) \right)^{1/2} \left(\int_V |x|^{s'-(n-1)}|\widehat{\mu_V^\delta}(x)|^2 \, d\mathcal{H}^{n-1}(x) \right)^{1/2} d\sigma V \\
&\leq \left(\int \int_V |x|^{t-(n-1)}|\widehat{\nu_V}(x)|^2 \, d\mathcal{H}^{n-1}(x) \, d\sigma V \right)^{1/2} \\
&\quad \times \left(\int \int_V |x|^{s'-(n-1)}|\widehat{\mu_V^\delta}(x)|^2 \, d\mathcal{H}^{n-1}(x) \, d\sigma V \right)^{1/2} \lesssim I_t(\nu)^{1/2}(1+I_s(\mu))^{1/2}.
\end{aligned} \tag{3.5}
$$

Next, to get the L^p-result we desired, we observe that functions $g \in L^q(G(n,n-1))$ with $\|g\|_{L^q} = 1$ (where q is dual to p) satisfy the kind of "power bound" as was required of σ. Namely,

$$
\int_{B(V,r)} g \, d\gamma_{n,n-1} \leq \gamma_{n,n-1}(B(V,r))^{1/p} \left(\int |g|^q \, d\gamma_{n,n-1} \right)^{1/q} \lesssim r^{(n-1)/p}.
$$

So, if $p > 1$ is so close to one that that $(n-1)/p \geq h$, the estimate (3.5) yields

$$
\begin{aligned}
\int f_\delta \cdot g \, d\gamma_{n,n-1} &\leq \int \left(\int_V |\widehat{\mu_V^\delta}(x)||\widehat{\nu_V}(x)| \, d\mathcal{H}^{n-1}(x) \right) g(V) \, d\gamma_{n,n-1}(V) \\
&\lesssim I_t(\nu)^{1/2}(1+I_s(\mu))^{1/2}.
\end{aligned}
$$

By the usual $L^p - L^q$ duality, this proves (ii).

We record two further standard facts: for $\gamma_{n,n-1}$ almost every $V \in G(n,n-1)$,

(iii) the measure $\pi_{V\sharp}\mu$ lies in the fractional Sobolev space $H^{(s-(n-1))/2}(V)$, and
(iv) the measure $\pi_{V\sharp}\nu$ has finite t-energy.

Fact (iv) follows immediately from (3.3) with $\sigma = \gamma_{n,n-1}$. Fact (iii) does not quite follow from (3.4) as stated above (because we assumed $s' < s$), but it does follow from the variant of (3.4), where $s' = s$ and $\sigma = \gamma_{n,n-1}$; this remains true, as can be proven easily via "integration in polar coordinates", see for instance (24.2) in [7]. This gives fact (iii).

Assume that $V \in G(n,n-1)$ is a plane such that (iii) and (iv) hold. Then, as observed already in [8] (or see Theorem 17.3 in the book [7]), the Hardy-Littlewood maximal function of $\pi_{V\sharp}\mu$ belongs to $L^1(\pi_{V\sharp}\nu)$, which implies that the functions $\pi_{V\sharp}\mu_\delta$ converge to a limit in $L^1(\pi_{V\sharp}\nu)$ both in $L^1(\pi_{V\sharp}\nu)$, and $\pi_{V\sharp}\nu$ almost everywhere (to see this, observe that $\pi_{V\sharp}\mu_\delta = \psi_\delta^V * \pi_{V\sharp}\mu$ for some approximate identity ψ_δ^V on V, because ψ was chosen radial). We denote the limit by $g_V \in L^1(\pi_{V\sharp}\nu)$, so that

$$
[\pi_{V\sharp}\mu_\delta](v) \to g_V(v) \text{ for } \pi_{V\sharp}\nu \text{ almost every } v \in V.
$$

Recalling from (ii) that the sequence $(f_\delta)_{\delta>0}$ is bounded in $L^p(G(n,n-1))$ for some $p > 1$, we may pick a subsequence $(f_{\delta_j})_{j\in\mathbb{N}}$, which converges weakly in $L^p(G(n,n-1))$

to a limit $f \in L^p(G(n, n-1))$. The values of f are known to us: since $\pi_{V\sharp}\mu_\delta \to g_V$ in $L^1(\pi_{V\sharp}\nu)$, whenever (iii) and (iv) hold (that is, for $\gamma_{n,n-1}$ almost every V), the whole sequence $(f_\delta)_{\delta>0}$ converges pointwise $\gamma_{n,n-1}$ almost everywhere, and we may infer that

$$f(V) = \int_V g_V \, d\pi_\sharp \nu \text{ for } \gamma_{n,n-1} \text{ almost every } V \in G(n, n-1).$$

On the other hand, since f is the weak L^p-limit of the functions f_{δ_j}, each of which satisfies the uniform L^1 lower bound from (i), we have

$$\|f\|_{L^1(G(n,n-1))} \geq \limsup_{j\to\infty} \|f_{\delta_j}\|_{L^1(G(n,n-1))} \geq c > 0.$$

This estimate is legitimate, because $G(n, n-1)$ is compact. It follows that $f(V) > 0$ for $\gamma_{n,n-1}$ positively many planes $V \in G(n, n-1)$.

Using this fact, we find a plane $V \in G(n, n-1)$ with the following four properties: (3.2), (iii) and (iv) hold, and

$$\int g_V \, d\pi_{V\sharp}\nu > 0. \tag{3.6}$$

The proof is finished by showing that the four conditions cannot, in fact, hold simultaneously. To this end, write

$$G_V := \{v \in V : [\pi_{V\sharp}\mu_\delta](v) \to g_V(v)\},$$

so that $\pi_{V\sharp}\nu(V \setminus G_V) = 0$, as we discussed after (iii) and (iv). Then, decompose the integral in (3.6) as follows:

$$\int g_V \, d\pi_{V\sharp}\nu = \int_{\{v\in G_V : \pi_V^{-1}\{v\}\cap B=\emptyset\}} \ldots + \int_{\{v\in G_V : \pi_V^{-1}\{v\}\cap B\neq\emptyset\}} \ldots + \int_{V\setminus G_V} \ldots.$$

The third integral is clearly zero, and the same is true for the second integral by (3.2):

$$\pi_{V\sharp}\nu(\{v : \pi_V^{-1}\{v\} \cap B \neq \emptyset\}) = \nu(\{x : L_{V,x} \cap B \neq \emptyset\}) = 0$$

But also the first integral is zero: indeed, if $v \in V$ and $\pi_V^{-1}\{v\} \cap B = \emptyset$, then the compactness of B implies that $\pi_V^{-1}(B(v, \delta)) \cap B = \emptyset$ for $\delta > 0$ small enough. If moreover $v \in G_V$, this implies that

$$g_V(v) = \lim_{\delta\to 0} [\pi_{V\sharp}\mu_\delta](v) = 0.$$

In other words, $g_V(v) = 0$ for every $v \in \{G_V : \pi_V^{-1}\{v\} \cap B = \emptyset\}$. We have now seen that (3.6) cannot hold, and the ensuing contradiction completes the proof.

3.2. **Sharpness of the bound.** Given $n-1 < s < n$, there exists a compact set $K \subset \mathbb{R}^n$ such that $\dim K = s$, and

$$\dim\{V \in G(n, n-1) : \mathcal{H}^{n-1}(\pi_V(K)) = 0\} = 2(n-1) - s. \tag{3.7}$$

The example is due to Peltomäki [9], but the details can also be found in [7], Example 5.13.

Consider the projective transformation $F \colon \mathbb{R}^n \setminus \mathbb{R}^{n-1} \to \mathbb{R}^n$, defined by

$$F(\tilde{x}, x_n) := \frac{(\tilde{x}, 1)}{x_n}, \qquad (\tilde{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \setminus \{0\}.$$

Then F maps lines in $\mathbb{R}^n$ of the form $\{te + (a,0) : t \in \mathbb{R}\}$, where $a \in \mathbb{R}^{n-1}$ and $e = (\tilde{e}, e_n) \in S^{n-1}$, $e_n \neq 0$, to lines of the form $\{u(a,1) + (\tilde{e}/e_n, 0) : u \in \mathbb{R}\}$. In particular, fixing the "base point" $a \in \mathbb{R}^{n-1}$, the mapping F takes the lines passing through $(a,0)$ to lines parallel to the vector $(a,1)$. Now, let $G := F^{-1}$, and consider the set $G(K) \subset \mathbb{R}^n$, which clearly still has $\dim G(K) = s$. The equation (3.7) can be (essentially) reworded by saying that there exists a $2(n-1) - s$ dimensional family E of vectors of the form $(a,1)$ such that K lies on an $\mathcal{H}^{n-1}$-null set of lines parallel to each $(a,1) \in E$. Hence, there exists a $2(n-1) - s$ dimensional family E' of points $a \in \mathbb{R}^{n-1}$ such that $G(K)$ lies on an $\mathcal{H}^{n-1}$-null set of lines passing through $(a,0)$. In other words, $\pi_{(a,0)}(G(K)) = 0$ for every $a \in E'$, as desired.

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